# Fault Tolerant Control

*A Simultaneous Stabilization Result*

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# Problem Formulation (1)

Consider a system of the form:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y_1 &= C_1 x \\ y_2 &= C_2 x \\ &\vdots \\ y_p &= C_p x \end{aligned}$$

Each of the $p$ measurements $y_i$, $i = 1, \ldots, p$, is the output of a sensor, which can potentially fail.

# Problem Formulation (2)

Consider a system of the form:

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\dot{x} &= Ax + Bu \\
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&\vdots \\
y_p &= C_p x
\end{aligned}
$$

Each of the $p$ measurements $y_i$, $i = 1, \ldots, p$, is the output of a sensor, which can potentially fail.

Does a *fixed* feedback compensator exist that stabilizes the system even in the faulty situations?

# Problem Formulation (3)

To be more precise, we are looking for a stabilizing dynamic compensator $u = K(s)y$ with the property, that the following feedback laws:

$$u = K(s)\begin{pmatrix} 0 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}, \ u = K(s)\begin{pmatrix} y_1 \\ 0 \\ \vdots \\ y_p \end{pmatrix}, \ \ldots, \ u = K(s)\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ 0 \end{pmatrix}$$

are also internally stabilizing, i.e. that both the nominal system as well as each of the systems resulting from one of the sensors failing are all stabilized by $K(s)$.

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- Problem Formulation
- **Preliminaries:**
  - The Diophantine Equation
  - Strong Stabilization
- Simultaneous Stabilization
- A Parameterization
- Determining the Parameters
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- Controller Structure
- Controller Order

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# The Diophantine Equation (1)

Let $M$ and $N$ be elements of a ring $\mathcal{L}$. Then $M$ and $N$ are coprime (i.e. $d \uparrow M, d \uparrow N \Rightarrow d^{-1} \in \mathcal{L}$) if and only if there exist $\tilde{U}, \tilde{V} \in \mathcal{L}$ such that

$\tilde{V}M + \tilde{U}N = u$, where $u$ is a unit, i.e. $u, u^{-1} \in \mathcal{L}$

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Example: let $\mathcal{L}$ be the ring of integers. Since, $14$ and $9$ are coprime, there exist $\tilde{V}$ and $\tilde{U}$, s.t.

$$14\tilde{V} + 9\tilde{U} = 1$$

One possible choice is: $\tilde{V} = 2$, $\tilde{U} = -3$.

# The Diophantine Equation (2)

Let $\mathcal{L}$ denote the ring of stable, proper, rational functions. Then $M$ and $N$ have no common zeros in the right half plane if and only if there exist $\tilde{U}, \tilde{V} \in \mathcal{L}$ s.t.

$$\tilde{V}M + \tilde{U}N = u \quad \text{where} \quad u, u^{-1} \in \mathcal{L}$$

i.e. $u$ is a stable minimum phase system of relative degree $0$.

# The Diophantine Equation (3)

Assume that two proper rational functions $G$ and $K$ are given. Then, it is always possible to choose $M, N, \tilde{V}, \tilde{U} \in \mathcal{L}$ such that $G = NM^{-1}$ and $K = \tilde{V}^{-1}\tilde{U}$. In that case, $K$ is an internally stabilizing compensator for $G$, if and only if

$$\tilde{V}M + \tilde{U}N = u \quad \text{where} \quad u, u^{-1} \in \mathcal{L}$$

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# Strong Stabilization

Let $A(s)$, $B(s)$ be stable proper transfer functions. Then there exists a stable proper transfer function $Q(s)$ such that the function

$$A(s) + B(s)Q(s)$$

is a unit in the ring of stable proper rational functions, if and only if the sign of

$$A(z_{\text{ip}})$$

is constant for all $z_{\text{ip}} \in \{s \in \mathcal{R}_{+\infty} \,:\, B(s) = 0\}$.

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# Simultaneous Stabilization (1)

Consider the following single-input dual-output system:

$$G = NM^{-1}(s) = \begin{pmatrix} N_1(s) \\ N_2(s) \end{pmatrix} M^{-1}(s)$$

(where $N, M \in \mathcal{L}$ are coprime) and assume that an internally stabilizing compensator is given:

$$K_0 = \tilde{V}_0^{-1}\tilde{U}_0 = \tilde{V}_0^{-1} \begin{pmatrix} \tilde{U}_{0,1} & \tilde{U}_{0,2} \end{pmatrix}$$

(where $\tilde{V}_0, \tilde{U}_0 \in \mathcal{L}$ are coprime) satisfying:

$$\tilde{V}_0 M - \tilde{U}_0 N = \tilde{V}_0 M - \tilde{U}_{0,1} N_1 - \tilde{U}_{0,2} N_2 = 1$$

# Simultaneous Stabilization (2)

If the sensor corresponding to one of the outputs fails, the controller $\tilde{V}^{-1}\begin{pmatrix} \tilde{U}_1 & \tilde{U}_2 \end{pmatrix}$ has to stabilize a system of the form:

$$G = \begin{pmatrix} N_1(s) \\ 0 \end{pmatrix} \qquad \text{or} \qquad G = \begin{pmatrix} 0 \\ N_2(s) \end{pmatrix}$$

In the first case, the Diophantine equation becomes

$$\tilde{V}M - \begin{pmatrix} \tilde{U}_1 & \tilde{U}_2 \end{pmatrix}\begin{pmatrix} N_1(s) \\ 0 \end{pmatrix} = \tilde{V}M - \tilde{U}_1 N_1 = u$$

# Simultaneous Stabilization (3)

In summary, a compensator $K = \tilde{V}^{-1}\tilde{U} = \tilde{V}^{-1}\begin{pmatrix} \tilde{U}_1 & \tilde{U}_2 \end{pmatrix}$ is fault tolerant if and only if

$$\begin{array}{ccccccc} \tilde{V}M & - & \tilde{U}_1 N_1 & - & \tilde{U}_2 N_2 & = & u_1 \\ \tilde{V}M & & & - & \tilde{U}_2 N_2 & = & u_2 \\ \tilde{V}M & - & \tilde{U}_1 N_1 & & & = & u_3 \end{array}$$

where $u_1, u_2, u_3$ are units. (WLOG, assume that $u_1 = 1$.)

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# A Parameterization (1)

A crucial observation is that if

$$K_0 = \tilde{V}_0^{-1} \begin{pmatrix} \tilde{U}_{0,1} & \tilde{U}_{0,2} \end{pmatrix}$$

is a stabilizing compensator for the nominal system,

# A Parameterization (1)

A crucial observation is that if

$$K_0 = \tilde{V}_0^{-1} \begin{pmatrix} \tilde{U}_{0,1} & \tilde{U}_{0,2} \end{pmatrix}$$

is a stabilizing compensator for the nominal system, so is

$$K = \tilde{V}^{-1} \begin{pmatrix} \tilde{U}_1 & \tilde{U}_2 \end{pmatrix}$$

where

$$\begin{aligned} \tilde{V} &= \tilde{V}_0 - Q_2 N_1 - Q_3 N_2 \\ \tilde{U}_1 &= \tilde{U}_{0,1} - Q_1 N_2 - Q_2 M \\ \tilde{U}_2 &= \tilde{U}_{0,2} + Q_1 N_1 - Q_3 M \end{aligned}$$

and $Q_1, Q_2, Q_3$ are arbitrary stable functions.

# A Parameterization (2)

$K = \tilde{V}^{-1} \begin{pmatrix} \tilde{U}_1 & \tilde{U}_2 \end{pmatrix}$ is seen to be nominally stabilizing since:

$$\begin{aligned} &\tilde{V}M - \tilde{U}_1 N_1 - \tilde{U}_2 N_2 \\ &= \left(\tilde{V}_0 - Q_2 N_1 - Q_3 N_2\right) M \\ &\quad - \left(\tilde{U}_{0,1} - Q_1 N_2 - Q_2 M\right) N_1 \\ &\quad - \left(\tilde{U}_{0,2} + Q_1 N_1 - Q_3 M\right) N_2 \\ &= \tilde{V}_0 M - \tilde{U}_{0,1} N_1 - \tilde{U}_{0,2} N_2 = 1 \end{aligned}$$

# Stability during faults (1)

This means that stability is obtained if and only if the compensator satisfies the two equations:

$$\tilde{V}_0 M - \begin{pmatrix} \tilde{U}_1 & \tilde{U}_2 \end{pmatrix} \begin{pmatrix} N_1 \\ 0 \end{pmatrix}$$

$$= \tilde{V}_0 M - Q_2 N_1 M - Q_3 N_2 M$$

$$- \tilde{U}_{0,1} N_1 + Q_1 N_2 N_1 + Q_2 M N_1$$

$$= \tilde{V}_0 M - \tilde{U}_{0,1} N_1 + Q_1 N_2 N_1 - Q_3 N_2 M = u_1$$

# Stability during faults (2)

and

$$\tilde{V}M - \begin{pmatrix} \tilde{U}_1 & \tilde{U}_2 \end{pmatrix} \begin{pmatrix} 0 \\ N_2 \end{pmatrix}$$

$$= \tilde{V}_0 M - \tilde{U}_{0,2} N_2 - Q_1 N_1 N_2 - Q_2 N_1 M = u_2$$

where $u_1, \ u_2$ are units in the ring of stable proper rational functions (i.e. stable proper functions with stable proper inverses).

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# Selecting $Q_1$

Note, that it is possible to determine a stable proper function $Q_1$, such that:

$$Q_1(s)N_1(s)N_2(s) - \tilde{U}_{0,1}(s)N_1(s)\Big|_{s=z_{\text{ip}}} = \frac{1}{2}$$

for every value of $z_{\text{ip}} \in \{z \in \mathcal{R}_{+\infty} \,:\, M(z) = 0\}$, since $N_1(z_{\text{ip}})N_2(z_{\text{ip}})$ can not be zero for $M(z_{\text{ip}}) = 0$ due to coprimeness of $M$ and $N_1$ and of $M$ and $N_2$. To determine $Q_1$ in practice can be done by a standard rational interpolation.

# Determining $Q_3$ (1)

Returning to the equation:

$$\tilde{V}_0 M - \tilde{U}_{0,1} N_1 + Q_1 N_2 N_1 - Q_3 N_2 M = u_1$$

For fixed $Q_1$ this can be solved by stable $Q_3$ if and only if (strong stabilization result)

$$\tilde{V}_0 M - \tilde{U}_{0,1} N_1 + Q_1 N_2 N_1 \Big|_{s=z_{\text{ip}}}$$

has constant sign for every value of $z_{\text{ip}} \in \{z \in \mathcal{R}_{+\infty} : M(z) = 0 \vee N_2(z) = 0\}$.

# Determining $Q_3$ (2)

For $M(z_{\text{ip}}) = 0$ we obtain:

$$\tilde{V}_0(s)M(s) - \tilde{U}_{0,1}(s)N_1(s) + Q_1(s)N_2(s)N_1(s)\Big|_{s=z_{\text{ip}}}$$

$$= \; -\tilde{U}_{0,1}(s)N_1(s) + Q_1(s)N_2(s)N_1(s)\Big|_{s=z_{\text{ip}}} = \frac{1}{2}$$

For $N_2(z_{\text{ip}}) = 0$ we get:

$$\tilde{V}_0(s)M(s) - \tilde{U}_{0,1}(s)N_1(s) + Q_1(s)N_2(s)N_1(s)\Big|_{s=z_{\text{ip}}}$$

$$= \tilde{V}_0(s)M(s) - \tilde{U}_{0,1}(s)N_1(s)\Big|_{s=z_{\text{ip}}} = 1$$

# Determining $Q_3$ (3)

To determine $Q_3$ in practice, one approach is first to find $u_1$ that interpolates all right half plane conditions (not just the positive half line) induced by $N_2$ and $M$. Then $Q_3$ can be computed by:

$$Q_3 = \frac{\tilde{V}_0 M - \tilde{U}_{0,1} N_1 + Q_1 N_2 N_1 - u_1}{N_2 M}$$

which is a stable proper solution to

$$\tilde{V}_0 M - \tilde{U}_{0,1} N_1 + Q_1 N_2 N_1 - Q_3 N_2 M = u_1$$

# Determining $Q_2$

Similar considerations regarding the equation

$$\tilde{V}_0 M - \tilde{U}_{0,2} N_2 - Q_1 N_1 N_2 - Q_2 N_1 M = u_2$$

proves the existence of a stable solution $Q_2$, e.g. in terms of the formula:

$$Q_2 = \frac{\tilde{V}_0 M - \tilde{U}_{0,2} N_2 - Q_1 N_1 N_2 - u_2}{N_1 M}$$

where $u_2$ has been chosen such that

$$u_2(z)$$

$$= \tilde{V}_0(z) M(z) - \tilde{U}_{0,2}(z) N_2(z) - Q_1(z) N_1(z) N_2(z)$$

for every $z \in \{z \in \mathcal{C}_+ : N_1(z) = 0 \vee M(z) = 0\}$.

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# Main Result (1)

**THEOREM.** Assume a system has the form:

$$\begin{aligned} \dot{x} &= Ax + Bu & (A, B) \text{ is stabilizable} \\ y_1 &= C_1 x & (C_1, A) \text{ is detectable} \\ y_2 &= C_2 x & (C_2, A) \text{ is detectable} \end{aligned}$$

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Then there exists a fault tolerant controller $K(s)$ such that

$$u = K \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \; u = K \begin{pmatrix} y_1 \\ 0 \end{pmatrix}, \; u = K \begin{pmatrix} 0 \\ y_2 \end{pmatrix},$$

are all internally stabilizing feedback laws.

# Main Result (2)

Moreover, one particular fault tolerant controller is given by:

$$K = \left(\tilde{V}_0 - Q_2 N_1 - Q_3 N_2\right)^{-1}$$
$$\times \left( \tilde{U}_{0,1} - Q_1 N_2 - Q_2 M \quad \tilde{U}_{0,2} + Q_1 N_1 - Q_3 M \right)$$

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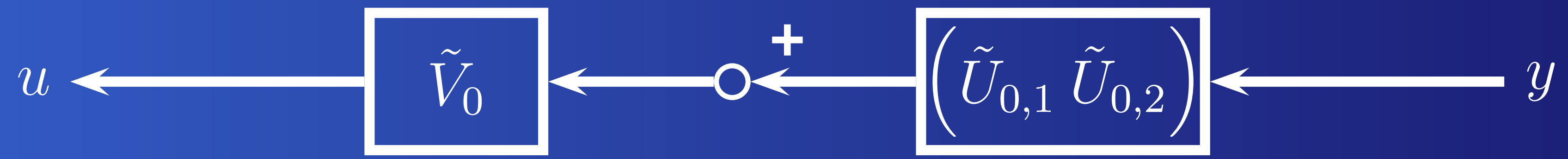
# Controller Structure



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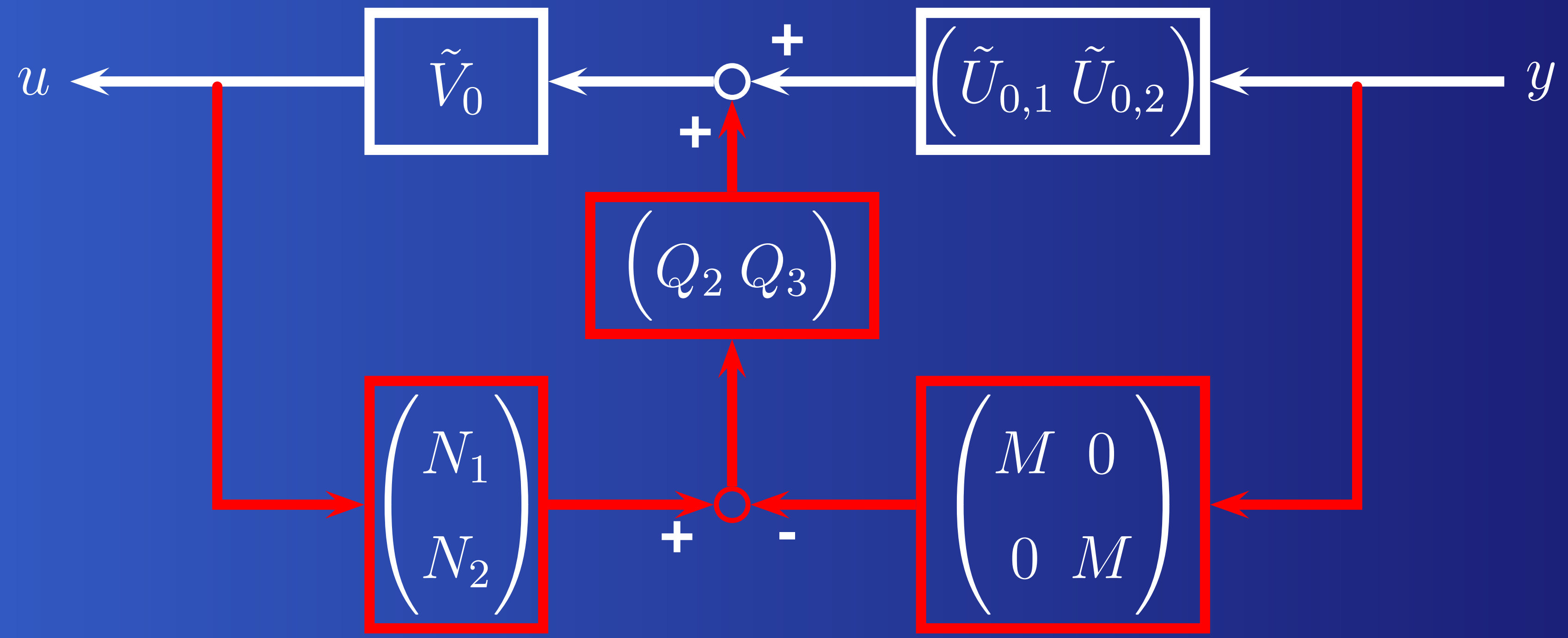

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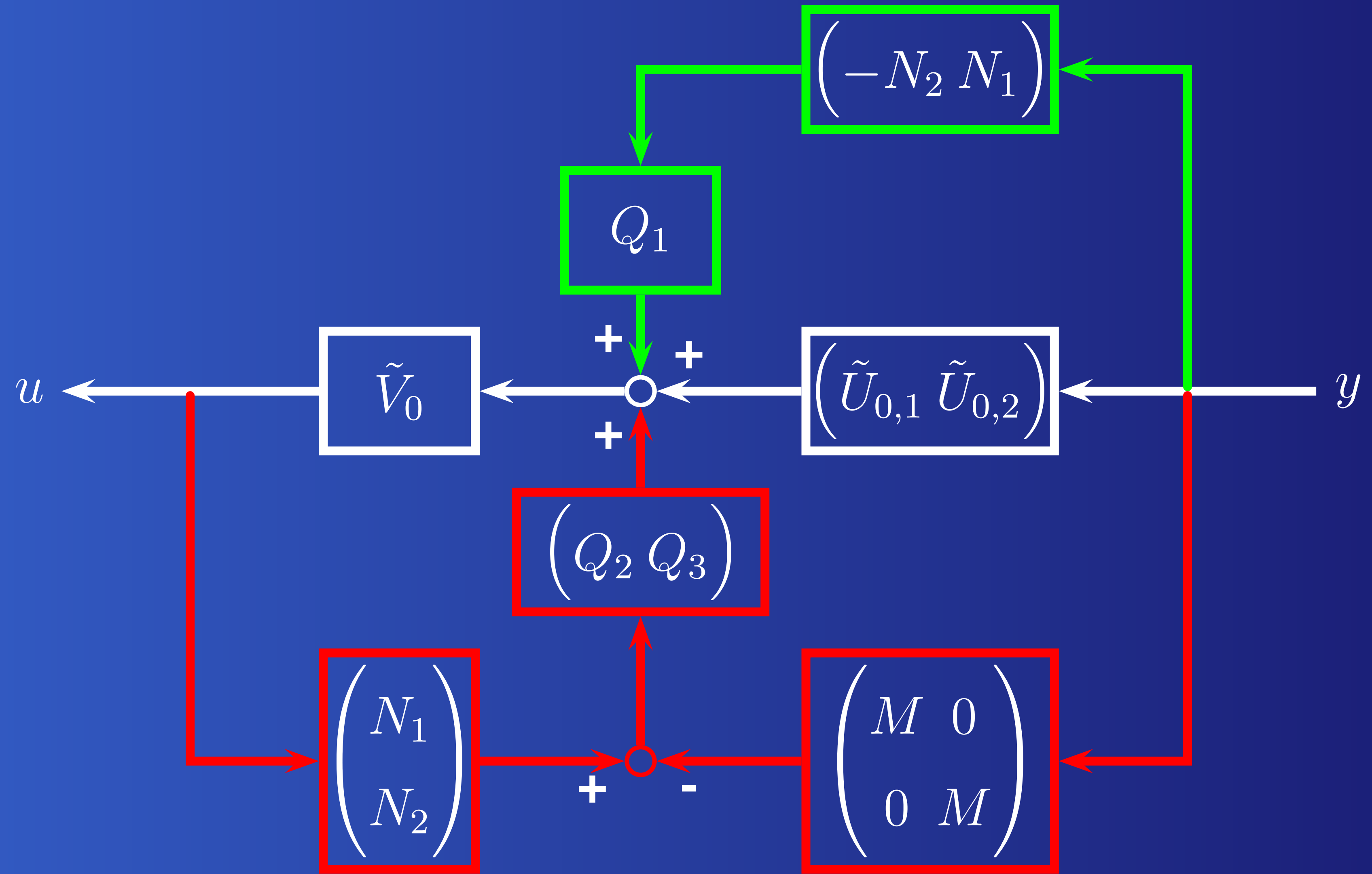

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# Controller Order

Consider the following family of systems for $\varepsilon > 0$:

$$G_\varepsilon(s) = \begin{pmatrix} \frac{s-1}{(s-(1+\varepsilon))(s+1)} \\ \frac{s-1}{(s-(1+\varepsilon))(s+1)} \end{pmatrix}$$

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It can be shown that the controller order of any fault tolerant controller for this system has to satisfy

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This means that the controller order tends to infinity as $\varepsilon$ tends to zero!